数理解析研究所講究録
第 1675 巻 2010 年 234-247

Optimal Execution Strategies with Price Impact *

大阪大学・経済学研究科 久納 誠矢 (Seiya Kunou) †
Graduate School of Economics
Osaka University
大阪大学・経済学研究科 & 金融保険教育研究センター 大西 匡光 (Masamitsu Ohnishi) ‡
Graduate School of Economics
& Center for the Study of Finance and Insurance
Osaka University

1 Introduction

This paper analyzes trading actions of a trader who executes large amount of a single risky asset. we call such trader large trader. In the competitive financial market paradigm, there is no quantity effect on the asset price. But in real situations, financial markets are not perfectly elastic. Then, large amount of trades can affect prices. If the large trader such as institutional trader submits large amount of his or her buy orders to the risky asset in the illiquid market, then the asset price would ascend because of the imbalance of supply and demand, and the price would fall when sell orders. Therefore, when the large trader determine his or her optimal trade schedules, he or she must realize these price fluctuations (impacts) as cost and pay attention to such impact risk. In addition, transparency and volatility of the market are also taken care of. Market liquidity have been studied by many researchers and practitioners. For example, in market microstructure literature, asymmetric information among the traders is considered as one of the cause of the liquidity risk (e.g. Kyle (1985)). Liquidity and volatility risk have been proposed and modeled variously, but as for transparency, it is difficult to evaluate clearly. In our model, impact is represented as λ based on Kyle (1985), volatility risk is represented as trading volume of the noise trader and public news effects to price.

The problem of optimal execution have developed from various viewpoints, for examples, the micro (static) strategy or the macro (dynamic) strategy, and the discrete time model or the continuous time model and so on. In the discrete time framework, Bertsimas and Lo (1998) consider the optimal purchase strategy of risk-neutral large trader for a single risky asset with the dynamic programming algorithm and show explicit optimal execution strategy. Almgren and Chriss (2000) extend the framework of Bertsimas and Lo (1998) with static mean-variance approach. They decompose price impact into temporary impact and permanent impact and reveal the static sell strategy for risk-averse large trader. Huberman and Stanzl (2005) also extend the work of Bertsimas and Lo (1998) and reveal the optimal strategy in static class. Our model is based on their framework, and we show their model in Section 2. Kissel and Malamut

*The authors would like to thank Jyunya Umeno, Michito Nakabayashi (BlackRock Japan) for their helpful comments. The responsibility of any errors is of course ours.

†E-mail: knuseya@gmail.com
‡E-mail: ohnishi@econ.osaka-u.ac.jp

(2006) propose several optimal execution strategies based on the framework of Almgren and Chriss (2000).

We determine execution strategies with Huberman and Stanzl (2005) framework in the dynamic class. Then, as stated Almgren (2000) and Almgren and Lorenz (2007), we show that the optimal execution trajectory exist in the static class and further, show that the optimal execution volume for each time is expressed as impact and volatility risk. If the trading speed of the large trader is slow, the large trader sustains the volatility risk because of the uncertainty of the future price. On the other hand, if the large trader executes more at the early stage, he or she sustains more of the impact risk because of large amount of his order. Therefore, we derive trading strategies which is balanced these risk: volatility and impact risk.

The paper is organized as follows. Section 2 presents model dynamics and discuss price impact briefly. Section 3 presents the optimal execution strategy with dynamic programming algorithm generally. In particular, Section 3.1 presents the result for risk-averse large trader, and Section 3.2, for risk-neutral large trader with the framework of Bertsimas and Lo (1998). Section 4 presents the properties of optimal execution volume with time-homogeneous impact and the reversion rate. Section 5 describes numerical examples using the framework in Section 4. Section 6 concludes the paper.

2 Market Model and Price Impact

In this section, we explain our market model and the price impact. For our market dynamics, we follow the discrete time framework proposed by Huberman and Stanzl (2005), which extends that of Bertsimas and Lo (1998) and genaralizes that of Kyle (1985). One risk averse large trader and many noise traders are considered as economic agents and implicitly we suppose that there exists a risk-neutral market maker. As follows, we analyze the optimal execution (purchase) strategy of the large trader for a single risky asset over time T who must purchase the predetermined total amount $\overline{Q}$ shares.

2.1 Dynamics

Suppose that p_t is the price of a single risky asset at time t, q_t is the large trader's execution volume for that risky asset, Q_t is the number of shares which the large trader remains to purchase, and w_t is investment capital (wealth). Because at time t the large trader submits large amount of his order q_t just after he has recognized the price at that time p_t, the execution time lag is formed because of the temporary imbalance of supply and demand. Then execution price represented as $\widehat{p}_t$ is not the price at time t but the price a little after. The empirical study of Almgren, Thum, Hauptmann, and Li (2005) demonstrated that execution time lags are within about 30 minutes though these depend on the volume of dealings and the thickness of the market. Based on this, in Section 5 we will demonstrate some intraday tradings that one period is assumed to be 30 minutes. Dynamics of the wealth and the remaining execution volume are

$$w_{t+1} = w_t - \widehat{p}_t q_t, \quad (2.1)$$
$$Q_{t+1} = Q_t - q_t. \quad (2.2)$$

The total order volume η_t submitted by the noise trader at time t is denoted as a random variable, and assume that it's small enough compared with the order volume of the large trader. Then execution price during the trading period is

$$\widehat{p}_t = p_t + \lambda_t(q_t + \eta_t). \quad (2.3)$$

Moreover, consider that the total order volume for that risky asset submitted by the large and the noise trader at time t is executed completely by the time $t+1$. λ_t is a variable of the price sensitivity per share and expresses the thickness of the market, denoted by Kyle (1985). The ascent of the price by the bulk purchase at time t is shown by the λ_t and the total order volume $q_t + \eta_t$. This is, so-called, the price impact. Now, suppose $\lambda_t \geq 0$ for all t, if the total order volume is positive then the execution price rises compared with the price at t, and similarly if negative then the execution price falls.

We assume the price at the next time $t+1$,

$$p_{t+1} = \alpha_t p_t + (1 - \alpha_t)\widehat{p}_t + \epsilon_{t+1}, \quad (2.4)$$

where α_t represents the reversion rate of price and follows $0 \leq \alpha_t \leq 1$. ϵ_{t+1} represents the public news effect to the price between time t and $t+1$ and is recognized by the large trader at time $t+1$.

$\{\eta_t\}_{t=1}^{T}$, $\{\epsilon_t\}_{t=2}^{T+1}$ are both i.i.d. stochastic processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and follow

$$\eta_t \sim N(\mu_\eta, \sigma_\eta^2); \quad \epsilon_t \sim N(\mu_\epsilon, \sigma_\epsilon^2). \quad (2.5)$$

Moreover, $\{\eta_t\}_{t=1}^{T}$ is independent of $\{\epsilon_t\}_{t=2}^{T+1}$.
All information available to the large trader before his trade at time t are

$$\mathcal{F}_t := \sigma\{(\eta_s, \epsilon_{s+1}) : s = 1, \dots, t-1\}. \quad (2.6)$$

Assume q_t to be $\mathcal{F}_t$ measurable real valued random variable that represents the large trader's volume of the order at time t, then the execution strategy can be represented as

$$\pi = (q_1, q_2, \dots, q_T). \quad (2.7)$$

Notice that, we can find from Equations (2.3), (2.4),

$$p_{t+1} = p_t + (1 - \alpha_t)\lambda_t(q_t + \eta_t) + \epsilon_{t+1}. \quad (2.8)$$

2.2 Price Impact

The price impact can be decomposed into the temporary impact and the permanent impact. The temporary impact represents the temporary imbalance between supply and demand, and affects only a present execution price. The permanent impact represents the price update and forms a new equilibrium price, which influences the price in the future (Almgren and Chriss (2000)).

Suppose $\epsilon = 0$. If α=1, then there is no price update from the previous period and price impact is equal to the temporary impact. On the other hand, if α=0, then the price impact of the present dealings is updated to the price of the following period. That is, price impact is equal to the permanent impact. Moreover, assume that we set α=0, η_t=0, then the this market model is equal to the market model of Bertsimas and Lo (1998). This means that the model of Bertsimas and Lo (1998) is simplified model which considers neither noise trader's submitting order nor the temporary impact.

One of the important problems of impact are related to the volume dependence and the temporal behaviour. Several empirical researches show that a concave impact function is usually observed (e.g., Almgren et al. (2005)). Also a concave impact function theoretically modeled various ways (e.g. Bouchaud (2009)). For example, the square-root raw are given in the BARRA price impact model with volume time.

Other important problems of impact are price manipulation that caused asymmetric impact for buy and sell trades. Several empirical studies find that the price impact of buy trades is larger than that of sell trades because of sell constraint (e.g. Chan and Lakonishok (1993)). Theoritically, Jarrow (1992) investigates whether a large trader can make profits from price manipulation in the continuous-time market, and gives sufficient condition that arbitrage opportunity is precluded. Huberman and Stanzl (2004) shows that permanent impact must be linear to rule out price manipulation, and defines quasi-arbitrage.

Our model does not consider volume dependence effect deeply because the purpose of this thesis is mainly to investigate the optimal strategy and to give the intuitive interpretation of optimal execution volume. Further, when $\mu_\eta = \mu_\epsilon = 0$, it is impossible to manipulate the price under the definition of Huberman and Stanzl (2004), however as we will show in Section 5, it is possible to manipulate when $\mu_\eta \neq 0$ or $\mu_\epsilon \neq 0$.

3 Optimal Execution

3.1 Execution Strategy for Risk-Averse Large Trader

Assumed a risk-averse large trader having CARA (Constant Absolute Risk Aversion) type utility. We consider the problem of the dynamic execution strategy that maximizes his expected utility from his wealth. Suppose, trading strategy of this large trader is $\pi = (q_1, q_2, \cdots, q_T)$, risk aversion coefficient is $R > 0$. Then we define his expected utility under the trading strategy π at time t as

$$V_t^\pi := \mathbb{E}_t^\pi \left[-\exp\{-Rw_{t+1}\} \cdot 1_{\{Q_{T+1}=0\}} + (-\infty) \cdot 1_{\{Q_{T+1}\neq 0\}} \right], \tag{3.1}$$

where $1_{\{\cdot\}}$ is indicator function. Moreover we define the optimal value function,

$$V_t := \operatorname*{esssup}_{\pi} V_t^{\pi}\ , \qquad t = 1, \cdots, T. \tag{3.2}$$

Where the subscript t of the expectation represents the condition under all information available to the large trader at time t.

Because of the Markov property of the dynamics and path independence of his utility at the final period, V_t is a function of (w_t, p_t, Q_t), and by the principle of optimality, the optimality equation (Bellman equation) becomes as,

$$V_t(w_t, p_t, Q_t) = \sup_{q_t \in \mathbb{R}} \mathbb{E}[V_{t+1}(w_{t+1}, p_{t+1}, Q_{t+1}) \mid w_t, p_t, Q_t, q_t]. \tag{3.3}$$

We solve the sequence of optimal execution volume which achieves V_1 from the final period T by backward induction in t.

Theorem 1 (Optimal Execution Volume) *The optimal execution volume of large trader at time t is represented as an affine function of the remaining execution volume Q_t of the large trader at the time. That is, using β_t which is the rate of the remaining execution volume to the large trader, and γ_t which is trading of noise trader and other peculiar market factors, we can represent,*

$$q_t^* = \gamma_t + \beta_t Q_t. \tag{3.4}$$

Then a deterministic execution strategy becomes optimal.

Remark 1 From Theorem 1, optimal execution volume at each time t depend only on the state at the time through the remaining execution volume Q_t, not on wealth w_t and asset price p_t. Since this remaining execution volume can be controlled determinately, the optimal execution strategy exists in the static class of the execution strategy. (Almgren and Chriss (2000)), (Almgren and Lorenz (2006))

Proof. [of Theorem 1]

The optimal value function at time $t = T+1$ follows,

$$V_{T+1}(w_{T+1}, p_{T+1}, Q_{T+1}) = \begin{cases} -\exp\{Rw_{T+1}\}, & for\ Q_{T+1} = 0, \\ -\infty, & for\ Q_{T+1} \neq 0. \end{cases} \tag{3.5}$$

Then, at time T, large trader purchases all of his remaining execution volume.Therefore, optimal value function at final period T is

$$\begin{aligned} V_T(w_T, p_T, Q_T) &= \mathbb{E}[V_{T+1}(w_{T+1}, p_{T+1}, Q_{T+1}) \cdot 1_{\{Q_{T+1}=0\}} + (-\infty) \cdot 1_{\{Q_{T+1} \neq 0\}} \mid w_T, p_T, Q_T] \\ &= \mathbb{E}[-\exp\{-R(w_T - \widehat{p}_T Q_T)\} \mid w_T, p_T, Q_T] \\ &= \mathbb{E}[-\exp\{-Rw_T + R(p_T + \lambda_T(Q_T + \eta_T))Q_T\} \mid w_T, p_T, Q_T] \\ &= -\exp\{-Rw_T + Rp_TQ_T + R\lambda_T Q_T^2\} \cdot \exp\{R\lambda_T \mu_\eta Q_T + \frac{R^2}{2}\lambda_T^2 \sigma_\eta^2 Q_T^2\} \\ &= -\exp\{-Rw_T + Rp_TQ_T\} \cdot \exp\{RA_TQ_T^2 + RB_TQ_T\}, \end{aligned} \tag{3.6}$$

where we define

$$A_T := \lambda_T + \frac{R}{2}\lambda_T^2\sigma_\eta^2, \tag{3.7}$$
$$B_T := \lambda_T\mu_\eta. \tag{3.8}$$

Next, when at time $t = T-1$,

$$V_{T-1}(w_{T-1}, p_{T-1}, Q_{T-1}) = \sup_{q_{T-1}\in\mathbb{R}} \mathbb{E}[V_T(w_T, p_T, Q_T) \mid w_{T-1}, p_{T-1}, Q_{T-1}]. \tag{3.9}$$

Hence, substitute Equation (3.6) for the expectation part of the right side of Equation (3.9),

$$\begin{aligned}
&\mathbb{E}[-\exp\{-Rw_T - p_TQ_T\}\cdot\exp\{RA_TQ_T^2 + RB_TQ_T\} \mid w_{T-1}, p_{T-1}, Q_{T-1}] \\
= &-\exp\left\{\begin{array}{l} -Rw_{T-1} + Rp_{T-1}Q_{T-1} + RC_{T-1}q_{T-1}^2 - R(D_{T-1} + F_{T-1}Q_{T-1})q_{T-1} \\ +R(A_T + \frac{R}{2}(1-\alpha_{T-1}^2)\lambda_{T-1}^2\sigma_\eta^2 + \frac{R}{2}\sigma_\epsilon^2)Q_{T-1}^2 + R(B_T + (1-\alpha_{T-1})\lambda_{T-1}\mu_\eta + \mu_\epsilon)Q_{T-1} \end{array}\right\},
\end{aligned} \tag{3.10}$$

where we define,

$$\left(\begin{array}{l} C_{T-1} := A_T + \alpha_{T-1}\lambda_{T-1} + \frac{R}{2}\alpha_{T-1}^2\lambda_{T-1}^2\sigma_\eta^2 + \frac{R}{2}\sigma_\epsilon^2, \\ D_{T-1} := B_T - \alpha_{T-1}\lambda_{T-1}\mu_\eta + \mu_\epsilon, \\ F_{T-1} := 2A_T - (1-\alpha_{T-1})\lambda_{T-1} - R\alpha_{T-1}(1-\alpha_{T-1})\lambda_{T-1}^2\sigma_\eta^2 + R\sigma_\epsilon^2. \end{array}\right. \tag{3.11}$$

Then by differentiating in the index of Equation (3.10), we get optimal execution volume,

$$q_{T-1}^* = \frac{D_{T-1} + F_{T-1}Q_{T-1}}{C_{T-1}}. \tag{3.12}$$

Substitute this q_{T-1}^* to Equation (3.10),

$$V_{T-1}(w_{T-1}, p_{T-1}, Q_{T-1}) = -e^{-Rw_{T-1}+Rp_{T-1}Q_{T-1}}\cdot e^{RA_{T-1}Q_{T-1}^2 + RB_{T-1}Q_{T-1}}\cdot e^{-R\frac{D_{T-1}^2}{4C_{T-1}}}, \tag{3.13}$$

where we define,

$$A_{T-1} := A_T + \frac{R}{2}(1-\alpha_{T-1})^2\lambda_{T-1}^2\sigma_\eta^2 + \frac{R}{2}\sigma_\epsilon^2 - \frac{F_{T-1}^2}{4C_{T-1}}, \tag{3.14}$$
$$B_{T-1} := B_T + (1-\alpha_{T-1})\lambda_{T-1}\mu_\eta + \mu_\epsilon. \tag{3.15}$$

In general, by calculating in a similar way recursively, we get at time t,

$$q_t^* = \frac{D_t + F_tQ_t}{2C_t}, \tag{3.16}$$

and

$$\left(\begin{array}{l} C_t := A_{t+1} + \alpha_t\lambda_t + \frac{R}{2}\alpha_t^2\lambda_t^2\sigma_\eta^2 + \frac{R}{2}\sigma_\epsilon^2, \\ D_t := B_{t+1} - \alpha_t\lambda_t\mu_\eta + \mu_\epsilon, \\ F_t := 2A_{t+1} - (1-\alpha_t)\lambda_t - R\alpha_t(1-\alpha_t)\lambda_t^2\sigma_\eta^2 + R\sigma_\epsilon^2, \end{array}\right. \tag{3.17}$$

$$\left(\begin{array}{l} A_t := A_{t+1} + \frac{R}{2}(1-\alpha_t)^2\lambda_t^2\sigma_\eta^2 + \frac{R}{2}\sigma_\epsilon^2 - \frac{F_t^2}{4C_t}, \\ B_t := B_{t+1} + (1-\alpha_t)\lambda_t\mu_\eta + \mu_\epsilon - \frac{D_tF_t}{2C_t}. \end{array} \right. \tag{3.18}$$

Therefore, the optimal execution volume at time t can be represented as,

$$q_t^* = \gamma_t + \beta_t Q_t, \tag{3.19}$$

where $\gamma_t = \frac{D_t}{2C_t}$, $\beta_t = \frac{F_t}{2C_t}$. □

Next we show the property of this solution briefly. Form Equation (3.17),

$$\beta_t = \frac{2A_{t+1} - (1-\alpha_t)\lambda_t - R\alpha_t(1-\alpha_t)\lambda_t^2\sigma_\eta^2 + R\sigma_\epsilon^2}{2A_{t+1} + 2\alpha_t\lambda_t + R\alpha_t^2\lambda_t^2\sigma_\eta^2 + R\sigma_\epsilon^2}, \tag{3.20}$$

$$\gamma_t = \frac{B_{t+1} - \alpha_t\lambda_t\mu_\eta + \mu_\epsilon}{2A_{t+1} + 2\alpha_t\lambda_t + R\alpha_t^2\lambda_t^2\sigma_\eta^2 + R\sigma_\epsilon^2}, \tag{3.21}$$

where,

$$\begin{aligned} 2A_{t+1} &= 2A_{t+2} + R(1-\alpha_{t+1})^2\lambda_{t+1}^2\sigma_\eta^2 + R\sigma_\epsilon^2 - \beta_{t+1}F_{t+1} \\ &= 2A_T + R\sum_{i=t+1}^{T-1}(1-\alpha_i)^2\lambda_i^2\sigma_\eta^2 + (T-t-2)R\sigma_\epsilon^2 - \sum_{i=t+1}^{T-1}\beta_iF_i, \\ B_{t+1} &= B_{t+2} + (1-\alpha_{t+1})\lambda_{t+1}\mu_\eta + \mu_\epsilon - \beta_{t+1}D_{t+1} \\ &= B_T + \sum_{i=t+1}^{T-1}(1-\alpha_i)\lambda_i\mu_\eta + (T-t-2)R\mu_\epsilon - \sum_{i=t+1}^{T-1}\beta_iD_i. \end{aligned} \tag{3.22}$$

A_t means a permanent effect of risk aversion of the large trader from $t+1$ to T, and B_t means permanent effect of the noise trader and public news from $t+1$ to T. Then, β_t represents the execution rate related only to the large trader at time t, and γ_t represents trading volume which is considered the effect of the other trader under the existence of price impact at time t. Therefore we find that the large trader should take into account the volatility risk and impact risk.

3.2 Optimal Strategy for Risk-Neutral Large Trader

This subsection provides optimal execution strategies for risk-neutral large trader derived by Bertsimas and Lo (1998). They derived dynamic optimal strategies of buy trades that minimize the expected cost of executing $\overline{Q}$ within T period. Compared with Huberman and Stanzl (2005) framework, their framework ignores temporary impact ($\alpha = 0$), and does not consider the order submitted by noise traders ($\eta_t = 0$). Moreover considered that $\lambda_t = \lambda$, $\mu_{\epsilon_t} = 0$. In their framework, the order submitted by the large trader at time t is executed at time $t+1$. Then the law of motion for p_t is expressed as

$$\widehat{p}_t = p_{t+1} = p_t + \lambda q_t + \epsilon_{t+1}. \tag{3.23}$$

The goal of the large trader is to minimize his or her expected execution cost $\mathbb{E}[\sum_{t=1}^{T} p_{t+1}q_t]$, subject to $\sum_{t=1}^{T} q_t = \overline{Q}$. At time t, applying the Bellman equation, the optimal value function of the two state variables is given by

$$V_t(p_t, Q_t) = \inf_{q_t \in \mathbb{R}} \mathbb{E}[p_{t+1}q_t + V_{t+1}(p_{t+1}, Q_{t+1}) \mid p_t, Q_t, q_t], \tag{3.24}$$

which relates the optimal value of the objective function at time t to its optimal value at time $t+1$. Then, by starting at the final T, applying Equations (3.23), (3.24), the optimal execution volume at time t is:

$$q_t^* = \frac{Q_t}{T-t+1}, \quad t = 1, \ldots, T. \tag{3.25}$$

Therefore, when price impact is assumed to be time-homogeneous, we find that risk-neutral large trader executes equally at each time.

4 Optimal Strategy under Time-Homogeneous Impact

In the previous section, we gave an optimal execution strategy with dynamic programming algorithm. The purpose of this section is to give the analysis of the optimal strategy and an intuitive description under some assumption.

Assumption 1 (*Noise Trader's Behavior and Public News*)
Assume that the order volume submitted by noise traders is 0 on the average, and the price react from public news is 0 in the mean. That is, $\mu_\eta = \mu_\epsilon = 0$

Under above Assumption 1, it is clear that $D_t = 0, B_t = 0$. Then,

$$q_t^* = \beta_t Q_t. \tag{4.1}$$

Assumption 2 (*Optimal Execution under Time Homogeneous Impact and Reversion Rate*)
Assume that we set time-homogeneous price impact and reversion rate of price,

$$\lambda_t = \lambda \text{ and } \alpha_t = \alpha. \tag{4.2}$$

In the followings, to make an interpritation of the optimal volume easy to understand, we discusses the properties of the optimal strategy by using above Assumptions 1 and 2, and q_t represents the optimal execution volume q_t^*.

In order to analyze the properties of the optimal execution volume, we examine the dependence of R and λ and α on, and we set β as a function of (R, λ, α). That is,

$$\beta_t(R, \lambda, \alpha) = \frac{2A_{t+1} - (1-\alpha)\lambda - R\alpha(1-\alpha)\lambda^2\sigma_\eta^2 + R\sigma_\epsilon^2}{2A_{t+1} + 2\alpha\lambda + R\alpha^2\lambda^2\sigma_\eta^2 + R\sigma_\epsilon^2}. \tag{4.3}$$

As follow, we briefly show about Remark 1 in the previous section with these assumption. For simplicity, we denote $\beta_t := \beta_t(R, \lambda, \alpha)$, $Q_t := Q_t(R, \lambda, \alpha)$.
From Equations (4.1) and (2.2), for all t,

$$\beta_t = \frac{q_t}{Q_t} = \frac{Q_t - Q_{t+1}}{Q_t} = 1 - \frac{Q_{t+1}}{Q_t}. \tag{4.4}$$

Then,

$$1 - \beta_t = \frac{Q_{t+1}}{Q_t}. \tag{4.5}$$

From Equation (4.5), we get

$$\overline{Q} = Q_1, Q_t = \left\{ \prod_{i=1}^{t-1} (1 - \beta_i) \right\} \overline{Q}, \ t = 2, \cdots, T, T+1. \tag{4.6}$$

Therefore, Q_t can be controlled determinately.

Theorem 2 (Monotone Decreasing Property) *Under Assumptions 1 and 2, the optimal execution volume decreases monotonously for time. That is,*

$$q_1^* \geq q_2^* \geq \cdots \geq q_T^*. \tag{4.7}$$

Proof. First of all, we describe the sufficient condition for monotonous decreasing for all t. q_{t+1} and q_t are represented as

$$\begin{aligned} q_{t+1} &= \beta_{t+1}(Q_t - q_t), \\ q_t &= \beta_t Q_t. \end{aligned} \tag{4.8}$$

Then,

$$\beta_{t+1} \leq \frac{\beta_t}{1 - \beta_t}. \tag{4.9}$$

It is shown as follows that Inequality (4.9) holds. From the Equation (3.19),

$$\beta_t = \frac{F_t}{2C_t}. \tag{4.10}$$

Moreover, from the Equation (3.17),

$$\begin{aligned} 2C_t &= 2A_{t+1} + 2\alpha\lambda + R\alpha^2\lambda^2\sigma_\eta^2 + R\sigma_\epsilon^2 \\ &= 2A_{t+2} + R(1-\alpha)^2\lambda^2\sigma_\eta^2 + R\sigma_\epsilon^2 - \frac{F_{t+1}^2}{2C_{t+1}} + 2\alpha\lambda + R\alpha^2\lambda^2\sigma_\eta^2 + R\sigma_\epsilon^2 \\ &= 2C_{t+1} + K - \frac{F_{t+1}^2}{2C_{t+1}}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} F_t &= 2A_{t+1} - (1-\alpha)\lambda - R\alpha(1-\alpha)\lambda^2\sigma_\eta^2 + R\sigma_\epsilon^2 \\ &= F_{t+1} + K - \frac{F_{t+1}^2}{2C_{t+1}}. \end{aligned} \tag{4.12}$$

Here, we put,

$$K := R(1-\alpha)^2\lambda^2\sigma_\eta^2 + R\sigma_\epsilon^2 \geq 0. \tag{4.13}$$

Following from Equations (4.10), (4.11), (4.12),

$$\begin{aligned} \frac{\beta_t}{1-\beta_t} - \beta_{t+1} &= \frac{F_t}{2C_t - F_t} - \frac{F_{t+1}}{2C_{t+1}} \\ &= \frac{K}{2C_{t+1} - F_{t+1}} \geq 0, \ (2C_{t+1} - F_{t+1} > 0). \end{aligned} \tag{4.14}$$

□

This theorem indicates that in our model, risk-averse large trader takes the action that avoids the volatility risk by executing at the early stage.

Next, we analyze the effect of risk aversion. Suppose R_a and R_b are the risk aversion coefficient of the large trader a and b.

Theorem 3 (Risk Aversion effect) *The more risk averse the large trader is, the earlier he executes. That is, for all t, if $R_a \geq R_b$, then*

$$(0 \leq)\ Q_t(R_a, \lambda, \alpha) \leq Q_t(R_b, \lambda, \alpha). \tag{4.15}$$

Proof. For simplicity, we denote $Q_t(R_i) := Q_t(R_i, \lambda, \alpha)$, $\beta_t(R_i) := \beta_t(R_i, \lambda, \alpha)$, $i = a, b$. $Q_t(R_a) \leq Q_t(R_b)$ is equivalent to $\beta_t(R_a) \geq \beta_t(R_b)$ because of Equation (4.1). Assume that

$$\beta_{t+1}(R_a) \geq \beta_{t+1}(R_b). \tag{4.16}$$

Then

$$\begin{aligned}
&\beta_t(R_a) - \beta_t(R_b) \\
&= \tfrac{F_t(R_a)}{2C_t(R_a)} - \tfrac{F_t(R_b)}{2C_{R_b}} \\
&= \frac{2A_{t+1}(R_a) - (1-\alpha)\lambda - R_a\alpha(1-\alpha)\lambda^2\sigma_\eta^2 + R_a\sigma_\epsilon^2}{2A_{t+1}(R_a) + 2\alpha\lambda + R_a\alpha^2\lambda^2\sigma_\eta^2 + R_a\sigma_\epsilon^2} - \frac{2A_{t+1}(R_b) - (1-\alpha)\lambda - R_b\alpha(1-\alpha)\lambda^2\sigma_\eta^2 + R_b\sigma_\epsilon^2}{2A_{t+1}(R_b) + 2\alpha\lambda + R_b\alpha^2\lambda^2\sigma_\eta^2 + R_b\sigma_\epsilon^2} \\
&= \frac{F_{t+1}(R_a) + R_a(1-\alpha)^2\lambda^2\sigma_\eta^2 + R_a\sigma_\epsilon^2 - \frac{F_{t+1}^2(R_a)}{2C_{t+1}(R_a)}}{2C_{t+1}(R_a) + R_a(1-\alpha)^2\lambda^2\sigma_\eta^2 + R_a\sigma_\epsilon^2 - \frac{F_{t+1}^2(R_a)}{2C_{t+1}(R_a)}} - \frac{F_{t+1}(R_b) + R_b(1-\alpha)^2\lambda^2\sigma_\eta^2 + R_b\sigma_\epsilon^2 - \frac{F_{t+1}^2(R_b)}{2C_{t+1}(R_b)}}{2C_{t+1}(R_b) + R_b(1-\alpha)^2\lambda^2\sigma_\eta^2 + R_b\sigma_\epsilon^2 - \frac{F_{t+1}^2(R_b)}{2C_{t+1}(R_b)}} \\
&= \frac{F_{t+1}(R_a) + L(R_a) - F_{t+1}\beta_{t+1}(R_a)}{2C_{t+1} + L(R_a) - F_{t+1}\beta_{t+1}(R_a)} - \frac{F_{t+1}(R_b) + L(R_b) - F_{t+1}\beta_{t+1}(R_b)}{2C_{t+1}(R_b) + L(R_b) - F_{t+1}\beta_{t+1}(R_b)},
\end{aligned}$$

where we define $L(R_i) := R_i(1-\alpha)^2\lambda^2\sigma_\eta^2 + R_i\sigma_\epsilon^2$, $i = a, b$. Therefore,

$$\begin{aligned}
\beta_t(R_a) - \beta_t(R_b) &\geq \frac{F_{t+1}(R_a) - F_{t+1}\beta_{t+1}(R_a)}{2C_{t+1} - F_{t+1}\beta_{t+1}(R_a)} - \frac{F_{t+1}(R_b) - F_{t+1}\beta_{t+1}(R_b)}{2C_{t+1}(R_b) - F_{t+1}\beta_{t+1}(R_b)} \\
&= \frac{\beta_{t+1}(R_a) - \beta_{t+1}^2(R_a)}{1 - \beta_{t+1}^2(R_a)} - \frac{\beta_{t+1}(R_b) - \beta_{t+1}^2(R_b)}{1 - \beta_{t+1}^2(R_b)} \\
&= \frac{\beta_{t+1}(R_a)}{1 + \beta_{t+1}(R_a)} - \frac{\beta_{t+1}(R_b)}{1 + \beta_{t+1}(R_b)} \\
&= \frac{1}{\frac{1}{\beta_{t+1}(R_a)} + 1} - \frac{1}{\frac{1}{\beta_{t+1}(R_b)} + 1} \\
&\geq 0,
\end{aligned} \tag{4.17}$$

where we use $L(R_a) \geq L(R_b)$, $R_a \geq R_b$, $\beta_{t+1}(R_a) \geq \beta_{t+1}(R_b)$. When $t = T-1$,

$$\begin{aligned}
\beta_{T-1}(R_a) &= \frac{F_{T-1}(R_a)}{2C_{T-1}(R_a)} \\
&= \frac{2\lambda + R_a\lambda^2\sigma_\eta^2(1-\alpha+\alpha^2) - (1-\alpha)\lambda + R_a\sigma_\epsilon^2}{2\lambda + R_a\lambda^2\sigma_\eta^2(1+\alpha^2) + 2\alpha\lambda + \sigma_\epsilon^2}.
\end{aligned} \tag{4.18}$$

Since $1-\alpha+\alpha^2 \geq 0$, it is obvious that $\beta_{T-1}(R_a) \geq \beta_{T-1}(R_b)$. Therefore,

$$Q_t(R_a) \leq Q_t(R_b),\ t=1,\cdots,T. \tag{4.19}$$

□

Remark 2 *Suppose $\lambda \neq 0$. As $R \downarrow 0$ implies*

$$\beta_t \to \frac{1}{T-t+1},\ t=1,2,\cdots,T. \tag{4.20}$$

while,as $R \to +\infty$ implies

$$\beta_t \to 1,\ t=1,2,\cdots,T. \tag{4.21}$$

This remark indicates that if $R \downarrow 0$, then the optimal execution strategy with price impact is equal to the naive strategy (Bertsimas and Lo (1998)). And if $R \to +\infty$, that is, the more risk averse the large trader is, the more he executes at early stage (Theorem 3).

Theorem 4 (No Price Impact) *If $\lambda = 0$, then $\beta_t = 1$, $t = 1,2,\cdots,T$, for any R and α, that is, if $\lambda = 0$, it is optimal for risk-averse large trader to execute at the initial time.*

Proof. It is obvious that $\beta_T(R,0,\alpha) = 1$, and $\frac{F_{T-1}}{2C_{T-1}} = \frac{R\sigma_\epsilon^2}{R\sigma_\epsilon^2} = 1$. In addition,

$$\beta_t(R,0,\alpha) = \frac{F_t}{2C_t} = \frac{2A_{t+1}+R\sigma_\epsilon^2}{2A_{t+1}+R\sigma_\epsilon^2} = 1,\ t=1,2,\cdots,T. \tag{4.22}$$

□

5 Numerical Examples

In this section, we illustrate optimal execution strategies for intraday trading. Trading time is based on NYSE (New York Stock Exchange), and we divide intraday into 13 periods as we stated in Section 2 (Almgren, Thum, Hauptmann, and Li (2005)). Assume that we must purchase 100,000 shares of risky asset within 13 periods. From the previous section, only one parameter is transformed for each figure and others are fixed. Through Figures 1, 3, 4, we set the parameter $\mu_\eta = \mu_\epsilon = 0$, $\sigma_\eta^2 = 1000$ and $\sigma_\epsilon^2 = 0.02$ (Huberman and Stanzl (2005)). Figures 1 and 2 are illustrated the dependence on risk aversion for the optimal execution strategy, where λ_t and α_t are fixed as $\lambda_t = 1.0 \times 10^{-5}$, $\alpha_t = 0.5$. In particular, when the case $\mu_\epsilon = 0.1$, we also illustrate in Figure 2. Moreover, we investigate the dependency of impact (in Figure 3) and reversion rate (in Figure 4) for the optimal execution strategy. In Figure 3, we set $R = 1.0 \times 10^{-8}$ and $\alpha_t = 0.5$. Further, in Figure 4, we set $R = 1.0 \times 10^{-8}$ and $\lambda_t = 1.0 \times 10^{-5}$.

Figure 1 illustrates Optimal execution strategies of the large trader having various risk aversion coefficients when $\mu_\eta = \mu_\epsilon = 0$ and β value. In order to avoid the volatility risk than the impact risk, risk-averse large trader executes more on the early stage. (Theorem 3).

Figure 2 illustrates that When $\mu_\eta = 0$ and $\mu_\epsilon = 0.1$, the less risk-averse large trader executes more on the early stage excessively in order to rise the price, and he sells the excess amount on the later stage. This indicates the manipulation.

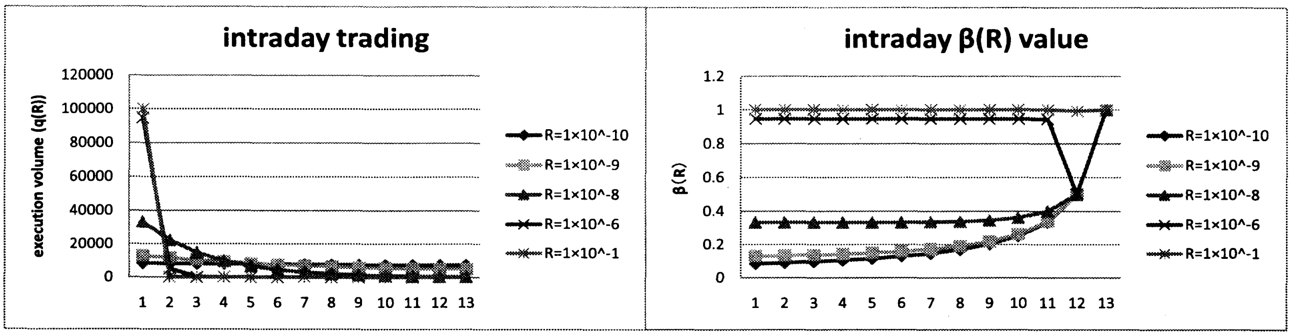

Figure 1: Dependence on risk aversion coefficients

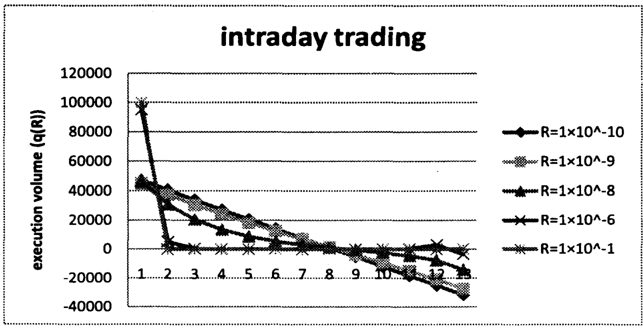

Figure 2: Dependence on risk avertion coefficients with positive public news

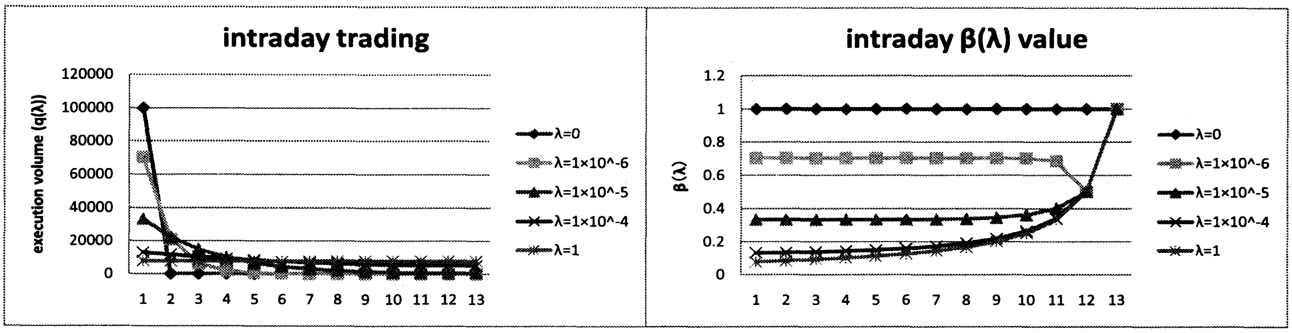

Figure 3: Dependence on various impacts

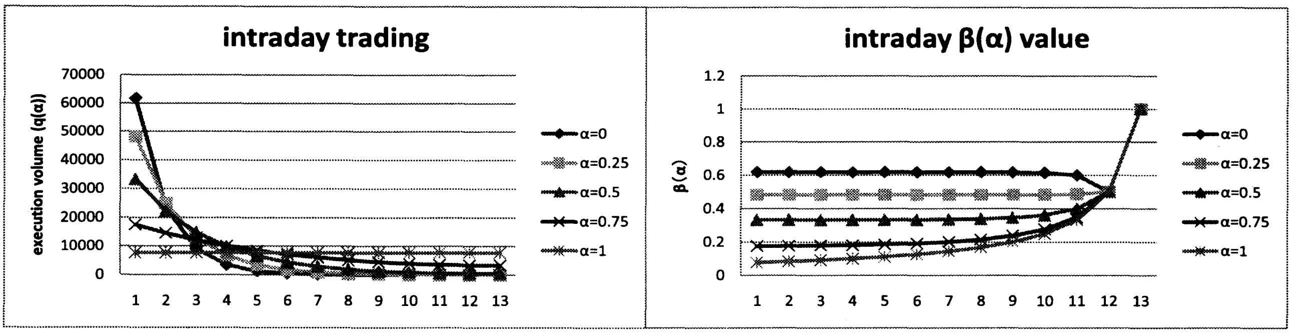

Figure 4: Dependence on risk reversion rates

Figure 3 illustrates optimal execution strategies of the large trader for various impacts and β value. When $\lambda_t = 0$, we find that it is optimal to executes 100,000 shares at the initial time (Theorem 4). When the value of λ is larger, it is optimal to execute equally for each time in order to avoid the impact risk.

Figure 4 shows optimal execution strategies of the large trader for various α and β value. When α come near to 1, it is optimal to execute equally for each time. That is, if the large amount of execution of the large trader less affect the price in the future, it is optimal to execute more equally. When the α value is smaller, the volatility risk is more trivial. Therefore, in this case, we find that it is optimal for the large trader to avoid impact risk.

The β values at time $t = 12$ above figures are $\frac{1}{2}$ until a certain level. One of the reason of this is because the value at the final period $t = 13$ is exactly 1. In other words, the β_t value (t=1,...,11) is influenced in the future β value, but β_{12} is only infuenced deterministic β value. (See Equations (3.11), (3.17), (3.20) and (3.22).)

6 Conclusion

This paper investigated properties of optimal execution volume with price impact derived from dynamic programming algorithm. We have obtained explicit solutions for the impact under mainly three special assumptions. First, the order of the noise trader and the public news effect to price are assumed to be normal random variables. As for more general distributions, we are remaining for future works. Second, we neglected the effect of impact for the trading volume. That is, we assumed both temporary and permanent impact to be linear for the trading volume. In particular, temporary impact is shown to be concave function of the trading volume by several empirical research. Therefore we have to establish the model of λ and α that is coincided with real markets. Third, we assumed time-homogeneous impact in Section 4. In many market microstructure literatures, it is reported that for liquidity, markets have peculiar shape, U-shape. That is, markets are time-inhomogeneous for liquidity. More sophisticated model considered time-inhomogeneous impact is also remaining for future works. But under these assumptions, we could obtain the solution that is easy to understand intuitively, and the relationship of strategy class. Moreover we analyzed the relationship between impact risk and volatility risk for the optimal execution strategy.

References

[1] Almgren, R. and N. Chriss: "Optimal execution of portfolio transactions," *Journal of Risk*, **3**, 5–39, 2000.

[2] Almgren, R., C. Thum E. Hauptmann, and H. Li: "Equity market impact," *Risk*, **18**, 57–62, 2005.

[3] Almgren, R. and J. Lorenz: "Adaptive Arrival Price," *Algorithmic Trading III: Precision, Control, Execution, Brian R. Bruce, editor, Institutional Investor Journals*, 2007.

[4] Bertsekas, D. P.: "Dynamic Programming and Stochastic Control," *Academic Press*, 1976.

[5] Bertsimas, D. and A. W. Lo: "Optimal control of execution costs," *Journal of Financial Markets*, **1**, 1–50, 1998.

[6] Bouchaud, J. P.: "Price Impact," *arXiv:0903.2428v1*, 2009.

[7] Brunnermeier, M. K.: "Asset Pricing under Asymmetric Information -Bubbles, Crashes, Technical Analysis, and Herding-," *Oxford University Press*, 2001.

[8] Chan, L. K. and J. Lakonishok: "Institutional trades and intraday stock price behavior," *Journal of Financial Economics*, **33**, 173–199, 1993.

[9] Huberman, G. and W. Stanzl: "Price manipulation and quasi-arbitrage," *Econometrica*, **72**, 1247–1275, 2004.

[10] Huberman, G. and W. Stanzl: "Optimal liquidity trading," *Review of Finance*, **9**, 165–200, 2005.

[11] Jarrow, G. A.: "Optimal liquidity trading," *Journal of Financial and Quantitative Analysis*, **27**, 311–336, 1992.

[12] Kissell, R. and R. Malamut: "Algorithmic decision making framework," *Journal of Trading*, **1**, 1, 12–21, 2006.

[13] Kyle, A.: "Continuous auctions and insider trading," *Econometrica*, **53**, 1315–1336, 1985.